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Note

An Efficient Algorithm for Calculating Thrust in High Multiplicity Reactions*

An efficient way of calculating the thrust variable is presented. With this method, the CPU goes up approximately as N^2, where N is the number of particles.

I. Introduction

The first order QCD matrix element for $e^+e^- \to q\bar{q}g$ diverges when the gluon is parallel to q or $\bar{q}$ or its energy is small. However, the cross section for events in which the fraction $1-\varepsilon$ $(\varepsilon \ll 1)$ of energy goes into back-to-back cones of half angle δ $(\delta \ll 1)$ is finite and reliably calculated [1]. This means that if a gluon is parallel to a quark, one cares about only the sum of the energies and not the fraction of energy carried by each particle. Thus, if a variable depends on that fraction of energy, that variable is not well defined in current QCD. Such a variable is called infra-red instable. For example, thrust is an infra-red stable variable, but sphericity is not. This is a big merit of thrust over sphericity.[1]

However, the difficulty of using thrust has been that the calculation time goes up as 2^N, where N is the number of particles, and it becomes impractical to calculate thrust for high multiplicity events. In the following sections the 2^N method will be shown to be correct; then its equivalence to the method which goes up as N^2 will be presented together with the algorithm.

II. 2^N Method

For any set of momentums $\{\mathbf{P}^i\}$ $i = 1,\dots, N$, the thrust T is defined as [5]

$$T = \frac{\max_{(\text{axis})} \sum_i |P^i_{\parallel}|}{\sum_i |P^i|}, \tag{1}$$

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[1] There exists a variable which is almost like sphericity and linear in momentum. One can use matrix diagonalization method for this variable and calculate a set of other variables just as in the case of Sphericity [2, 3]. Fox–Wolfram moment are also infra-red stable and easy to calculate [4].

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where $P^i_{||}$ is the component along a certain axis and the maximum is taken over all the orientations of the axis.

The problem is to find an axis which maximizes $\sum_i |P^i_{||}|$. To do this, one takes a set of signs $\{s_i\}$ $i = 1,\ldots, N$ $(s_i = +1$ or $-1)$ and forms $|\sum_i s_i \mathbf{P}^i|$ for all sign combinations $\{s_i\}$. Then the following equation holds:

$$\max_{\{s_i\}} \left| \sum_i s_i \mathbf{P}^i \right| = \max_{(\text{axis})} \sum_i |P^i_{||}|. \tag{2}$$

Proof. First, let $\{s_i^0\}$ be the set of signs which maximizes $|\sum_i s_i \mathbf{P}^i|$. Then one can show that

$$\max_{\{s_i\}} \left| \sum_i s_i \mathbf{P}^i \right| \equiv \left| \sum_i s_i^0 \mathbf{P}^i \right| = \sum_i |P^i_{||}|,$$

where $P^i_{||}$ is the component along the vector $\sum_i s_i^0 \mathbf{P}^i$ $(\equiv \mathbf{P}^0)$. Namely, for $\{s_i^0\}$, each momentum is contributing positively to $|\sum_i s_i^0 \mathbf{P}^i|$. The reason is that if there is any $\mathbf{P}^i$ which gives $\mathbf{P}^0 \cdot \mathbf{P}^i < 0$, then reversing the sign of that vector gives a longer $\sum_i s_i \mathbf{P}^i$, which contradicts with the assumption that the set $\{s_i^0\}$ gives maximum length.

Therefore,

$$\max_{\{s_i\}} \left| \sum_i s_i \mathbf{P}^i \right| = \sum_i |P^i_{||}| \leqslant \max_{(\text{axis})} \sum_i |P^i_{||}|. \tag{3}$$

where the parallel component is taken along $\mathbf{P}^0$ $(\equiv \sum_i s_i^0 \mathbf{P}^i)$ in the expression in the middle.

On the other hand, for the axis which maximizes $\sum_i |P^i_{||}|$ (namely, the thrust axis), we form the particular set of signs $\{s_i^T\}$ which satisfies the equation

$$\sum_i |P^i_{||_T}| = \sum_i s_i^T P^i_{||_T} \equiv \left(\sum_i s_i^T \mathbf{P}^i \right)_{||_T},$$

where s_i^T is defined by $|P^i_{||_T}| = s_i^T P^i_{||_T}$, and the symbol $||_T$ indicates that the parallel components are taken along the thrust axis.

Then, since $\{s_i^T\}$ is just a special case of $\{s_i\}$,

$$\max_{(\text{axis})} \sum_i |P^i_{||}| \equiv \sum_i |P^i_{||_T}| = \left(\sum_i s_i^T \mathbf{P}^i \right)_{||_T} \leqslant \left| \sum_i s_i^T \mathbf{P}^i \right| \leqslant \max_{\{s_i\}} \left| \sum_i s_i \mathbf{P}^i \right|. \tag{4}$$

From (3) and (4), (2) is proven. Thus, if one takes all the combination $\{s_i\}$ and finds a set $\{s_i^0\}$ which maximizes $|\sum_i s_i \mathbf{P}^i|$, then the thrust is given by

$$T = \frac{|\sum_i s_i^0 \mathbf{P}^i|}{\sum_i |P^i|}, \tag{5}$$

and the thrust axis is along

$$\sum_i s_i^0 \mathbf{P}^i. \tag{6}$$

With this method, the number of possible sign combinations $\{s_i\}$ grows as 2^N, and when the number of particles becomes more than ~ 20, the calculation becomes practically impossible.

III. N^2 Method

Actually, one does not have to take all sign combinations. For example, if any 3 vectors are assigned + signs, then any vector inside the back-to-back triangular cones formed by 3 lines along these 3 vectors has its sign already determined. As shown below, one can restrict oneself to "plane-separable" subsets of $\{s_i\}$ (Fig. 1). Namely, if one takes a plane that goes through the origin, and assigns + to vectors on one side of the plane and − to others, one gets a subset of general $\{s_i\}$, which we denote by $\{s_i'\}$. Since $\{s_i^T\}$ in Section II is obviously particular case of $\{s_i'\}$, formula (4) holds as well for $\{s_i'\}$, i.e.,

$$\max_{\{s_i'\}} |s_i \mathbf{P}^i| = \max_{(\text{axis})} |P_{||}^i|.$$

To count the ways the plane separates the momenta, take the normal vector $\mathbf{n}$ to that plane at the origin, and assume that one assignes + to $\mathbf{P}^i$ if $\mathbf{n}\cdot\mathbf{P}^i > 0$, and − if $\mathbf{n}\cdot\mathbf{P}^i < 0$. Or equivalently, for a vector $\mathbf{P}^i$ which is fixed in space, if $\mathbf{n}$ is in the hemisphere of $\mathbf{n}\cdot\mathbf{P}^i > 0$, then $\mathbf{P}^i$ is assigned +. This way the unit sphere of $\mathbf{n}$ can be divided into two hemispheres by a circle C_i with its direction defined right-handedly, i.e., if $\mathbf{n}$ is to the "left" of C_i, $\mathbf{P}^i$ is assigned + (Fig. 2).

If one draws N grand circles for N particles, there will be many patches on the unit sphere divided by these lines. Figure 3 shows an example for $N = 3$. Since $\mathbf{n}$ has to cross a line to change the sign of any particle, the sign combination is well defined and unique as long as $\mathbf{n}$ stays in a single patch. Thus, each patch uniquely corresponds to each plane-separable sign combination.

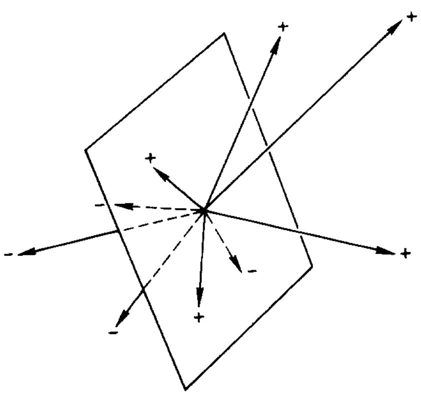

Figure 1

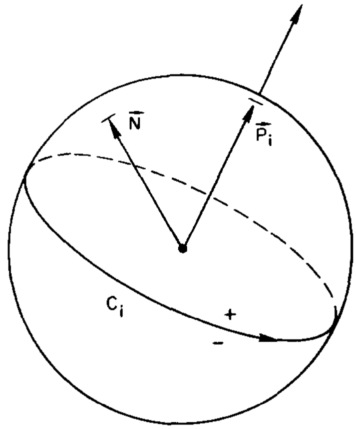

FIGURE 2

How many patches are there? This itself is an interesting mathematical problem, but it is easy to set an upper limit. Since each of the N rings intersects with $N-1$ other rings at two points, there are $2N(N-1)$ points on the sphere. Each point contributes to four patches and generally each patch needs at least three points, so that the upper limit is

$$\tfrac{8}{3}N(N-1).$$

Note that these patches are paired. For a patch there is a patch of exactly the same shape on the opposite side of the sphere. This corresponds to reversing the sign of all momenta. Thus, the upper limit of independent patches is one half of the above number, i.e.,

$$\tfrac{4}{3}N(N-1).$$

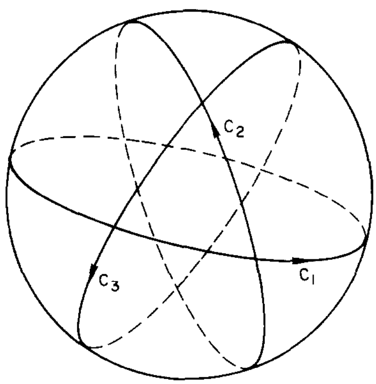

FIGURE 3

Now having shown that the number of sign combinations actually grows as N^2, we have to go through every sign combination to actually calculate the thrust variable. Since it is much easier to find intersections than patches themselves, let us start with intersecting points.

If $\mathbf{n}$ is on the intersection of C_i and C_j, everything else except $\mathbf{P}_i$ and $\mathbf{P}_j$ have well-defined signs. The direction of the intersection is given by $\mathbf{P}^i \times \mathbf{P}^j$, and $\mathbf{P}^k$ $(k \neq i, j)$ is assigned the sign of $\mathbf{P}^k \cdot (\mathbf{P}^i \times \mathbf{P}^j)$. Note that taking only one of 2 intersections of C_i, C_j is enough. For $\mathbf{P}^i$ and $\mathbf{P}^j$, all possible combinations of signs are taken. This makes 4 combinations $\{s_i\}$ for each independent intersection.

This way one exhausts all patches, when one goes through all intersections. But there are multiple countings. A triangular patch will be counted 3 times, a quadrangular patch 4 times, etc. However, this is not as inefficient as it looks, since when one calculates $\sum_i s_i \mathbf{P}^i$ for 4 sign combinations of (i, j), one does not have to recalculate except for momentum i and j.

The algorithm is very simple, and is as follows:

Loop over all combinations (i, j) with $i > j$, and each time set signs s_k $(k \neq i, j)$ to be the sign of $\mathbf{P}^k \cdot (\mathbf{P}^i \times \mathbf{P}^j)$ and for s_i, s_j take all 4 combinations. For each set of sign combinations, calclate $|\sum_i s_i \mathbf{P}^i|$ and check the maximum.

If $\{s_i^0\}$ gives the maximum, the thrust and its axis are given by (5), (6), respectively.

The algorithm itself can be arrived at more directly, viewing that any plane which separates the momentum vectors into two groups can be moved without changing any sign—namely, without crossing any momentum vector—to the state that it is almost touching two momentum vectors, and this state is obviously covered by the algorithms described above.

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